*Journal of Statistical Physics, Vol. 57, Nos. 3/4, 1989*

# Sound Modes in a Nonequilibrium Fluid with Acoustic Boundary Conditions

**R. Schmitz**[1]

*Received March 30, 1989*

The sound modes in a plane fluid layer with sound-absorbing walls at different temperatures are evaluated. The eigenvalue problem is solved by means of a singular perturbation theory and the WKB method. The bending of sound, the nonexponential damping, and the role of the wall admittance are discussed.

**KEY WORDS:** Hydrodynamic modes; nonequilibrium steady state; singular perturbation theory; WKB method; sound bending; sound damping; acoustic admittance.

## 1. INTRODUCTION

Hydrodynamic modes are eigensolutions of the hydrodynamic equations. As such, they find wide applications not only in fluid mechanics (solutions of initial value problems), but also in statistical mechanics. For example, the wavevector- and frequency-dependent intensity spectrum of light scattered from fluids can be understood on the basis of the interactions of the incoming light beam with certain hydrodynamic modes.[1]

For unbounded fluids in thermal equilibrium the modes are characterized by a wavevector **q**, and for each **q** there are two viscous modes, a heat mode, and two sound modes. If **q** is not too large, the viscous and the heat modes are purely diffusive, while the sound modes are propagating with weak damping. These small-**q** equilibrium results for unbounded fluids have, of course, long been well established. However, generalizations into various directions have been considered more recently.

One interesting option is to extend the hydrodynamic modes to large

[1] Department of Physics, University of Florida, Gainesville, Florida 32611.

$\mathbf{q}$ (being probed in neutron scattering), where standard hydrodynamics breaks down. These modes have been computed by de Schepper and Cohen[2] using kinetic theory. Staying within the small-$\mathbf{q}$ (i.e., hydrodynamic) regime, another possibility is to consider fluids in *non*equilibrium steady states. These states are established by imposing some constraints on the boundaries of the system which prevent it from relaxing to equilibrium. In this way one may induce a permanent shear stress or a heat flux in the fluid. The light scattering spectrum from fluids in nonequilibrium steady states shows exciting new features[3] which are most naturally explained in terms of the nonequilibrium hydrodynamic modes.[4]

In this paper I shall discuss the sound modes in a fluid layer bounded by two parallel plates of different temperatures. The resulting heat flux in the fluid modifies the sound modes in two ways: it causes sound bending (i.e., propagation along curved rays) and anomalous (i.e., nonexponential) sound damping. These effects occur because the speed of sound and the sound damping coefficient depend on the local temperature, which is a function of position.

A sound wave that propagates in the direction in which the local sound damping coefficient decreases actually experiences much less damping than in equilibrium. Boundary effects, therefore, cannot be ignored in many applications, even for large systems. For this reason I have computed the modes in the finite system, choosing so-called acoustic boundary conditions at the walls. These involve a parameter $\beta$, the specific acoustic admittance, which is a measure of the elasticity of the walls. Depending on the value of $\beta$, the walls are more or less sound-absorbing.

Since the full eigenvalue problem in the presence of a heat flux and acoustic boundary conditions is rather complex, I have chosen a stepwise presentation. In Section 2 I review some of the general equations and describe the properties of the sound modes in an unbounded equilibrium fluid. Then, in Sections 3 and 4, I discuss, still for the equilibrium case, the effect of the walls on the computation of the modes. Whereas in the bulk fluid the sound modes are primarily due to pressure and longitudinal velocity excitations, there is also an entropy component close to the walls. To determine the sound modes in the presence of the walls, I use a singular perturbation theory. Sections 5 and 6 deal with the further complications due to the heat flux. It is found that the behavior in the boundary layers is essentially the same as in equilibrium. However, in the bulk region, the position dependence of the sound speed and of the sound damping coefficient has to be taken into account. The solutions are obtained by means of the WKB method (Section 6). The complete expressions for the sound modes are presented in Section 7. Finally, in Section 8, I conclude with a discussion of the results.

Some of the results to be derived here have been applied earlier in work[5] on the structure of the Brillouin lines in fluids subject to a heat flux.

## 2. THE HYDRODYNAMIC MODES FOR AN UNBOUNDED FLUID IN EQUILIBRIUM

As an introduction to the subject, I review in this section the calculation of the hydrodynamic modes for an unbounded fluid in thermal equilibrium. To this end, I first write down the hydrodynamic equations, linearized about equilibrium:

$$\begin{aligned}
\frac{\partial}{\partial t}\delta p &= (\gamma-1)\,D_T\nabla^2\,\delta p - c\nabla\cdot\delta\mathbf{u} + (\gamma-1)^{1/2}\,D_T\nabla^2\,\delta s \\
\frac{\partial}{\partial t}\delta\mathbf{u} &= -c\nabla\,\delta p + (\Gamma_l - \nu)\,\nabla\nabla\cdot\delta\mathbf{u} + \nu\nabla^2\,\delta\mathbf{u} \qquad (2.1) \\
\frac{\partial}{\partial t}\delta s &= (\gamma-1)^{1/2}\,D_T\nabla^2\,\delta p + D_T\nabla^2\,\delta s
\end{aligned}$$

where the variables $\delta p(\mathbf{r}, t)$, $\delta\mathbf{u}(\mathbf{r}, t)$, and $\delta s(\mathbf{r}, t)$ denote the local deviations of the pressure, the flow velocity, and the entropy density, respectively, from their equilibrium values. These variables are scaled in (2.1), and throughout this paper, such that they all have the same dimension.[2] Furthermore, (2.1) involves as parameters the speed of sound $c$, the ratio $\gamma = c_p/c_v$ of the specific heats at constant pressure ($c_p$) and constant volume ($c_v$), and the generalized diffusion coefficients $\nu$, $\Gamma_l$, and $D_T$, denoting the kinematic viscosity, the longitudinal viscosity, and the thermal diffusivity, respectively. The generalized diffusion coefficients are proportional to the transport coefficients.

In formal notation, eqs. (2.1) can be written as

$$\frac{\partial}{\partial t}\delta\mathbf{a} = -\mathscr{H}\cdot\delta\mathbf{a} \qquad (2.2)$$

where $\delta\mathbf{a} = (\delta p, \delta\mathbf{u}, \delta s)$ and $\mathscr{H}$ is a linear hydrodynamic operator that follows from (2.1). Of course, this holds also when the system is bounded,

[2] The relations of $\delta p$, $\delta\mathbf{u}$, and $\delta s$ to the true deviations in pressure, velocity, and entropy per unit mass, denoted by $\delta p'$, $\delta\mathbf{u}'$, $\delta s'$, respectively, are given by[4]

$$\delta p' = (\gamma/\chi_T)^{1/2}\,\delta p, \qquad \delta\mathbf{u}' = \rho^{-1/2}\,\delta\mathbf{u}, \qquad \delta s' = (c_p/\rho T)^{1/2}\,\delta s$$

where $\rho$, $T$, and $\chi_T$ are the density, temperature, and isothermal compressibility, respectively, while $\gamma$ and $c_p$ are explained in the text.

in which case the equations have to be supplemented by boundary conditions, however. When perturbations about a nonequilibrium steady state are considered, one still obtains equations of the form (2.2), but with a more complicated operator $\mathscr{H}$. In any case we will be dealing here with evolution equations of the form (2.2). They are associated with the eigenvalue problem

$$\mathscr{H} \cdot \mathbf{a}_K(\mathbf{r}) = h_K \mathbf{a}_K(\mathbf{r}) \tag{2.3}$$

In (2.3) the index $K$ labels the modes, $h_K$ being an eigenvalue and $\mathbf{a}_K(\mathbf{r})$ the corresponding eigenvector. The eigenvectors may be normalized such that

$$\int d^3r\, \bar{\mathbf{a}}_K(\mathbf{r}) \cdot \mathbf{a}_{K'}(\mathbf{r}) = \delta_{KK'} \tag{2.4}$$

where $\bar{\mathbf{a}}_K(\mathbf{r})$ is the adjoint eigenvector of $\mathbf{a}_K(\mathbf{r})$, which solves the adjoint eigenvalue problem

$$\bar{\mathscr{H}} \cdot \bar{\mathbf{a}}_K(\mathbf{r}) = h_K \bar{\mathbf{a}}_K(\mathbf{r}) \tag{2.5}$$

where $\bar{\mathscr{H}}$ is the adjoint operator of $\mathscr{H}$ [with respect to the scalar product associated with (2.4)]. Assuming that the modes form a complete system, one can express the solution of the evolution equations (2.2) for given initial conditions $\delta \mathbf{a}(\mathbf{r}, 0)$ at $t = 0$ in the form

$$\delta \mathbf{a}(\mathbf{r}, t) = \sum_K \delta\alpha_K \mathbf{a}_K(\mathbf{r})\, e^{-h_K t} \qquad (t > 0) \tag{2.6}$$

where

$$\delta\alpha_K = \int d^3r\, \delta \mathbf{a}(\mathbf{r}, 0) \cdot \bar{\mathbf{a}}_K(\mathbf{r}) \tag{2.7}$$

After these formal remarks, let us turn to the solution of the eigenvalue problem (2.3), with $\mathscr{H}$ given by (2.1), for the unbounded case. Making use of translational invariance, it follows that the eigenvectors are of the form

$$\mathbf{a}_K(\mathbf{r}) = \hat{\mathbf{a}} \exp(i\mathbf{q} \cdot \mathbf{r}) \tag{2.8}$$

where $\mathbf{q}$ can be any (real) wavevector. Furthermore, it is advantageous to decompose the velocity $\hat{\mathbf{u}}$ into a longitudinal and a transverse part,

$$\hat{\mathbf{u}} = \hat{\mathbf{q}} \hat{u}^{(l)} + \hat{\mathbf{u}}^{(t)} \tag{2.9}$$

where $\hat{\mathbf{q}} = \mathbf{q}/q$ $(q = |\mathbf{q}|)$, and

$$\hat{u}^{(l)} = \hat{\mathbf{q}} \cdot \hat{\mathbf{u}}, \qquad \hat{\mathbf{u}}^{(t)} = (\mathbf{1} - \hat{\mathbf{q}}\hat{\mathbf{q}}) \cdot \hat{\mathbf{u}} \tag{2.10}$$

Inserting (2.8)–(2.10) into the eigenvalue equations, one immediately observes that $\mathbf{u}^{(t)}$ is decoupled from the other variables and satisfies the equation

$$\nu q^2 \hat{\mathbf{u}}^{(t)} = h \hat{\mathbf{u}}^{(t)} \tag{2.11}$$

This gives rise to two viscous modes (since there are two independent components of $\mathbf{u}^{(t)}$) which have both the same eigenvalue, namely

$$h_{\nu_1 \mathbf{q}} = h_{\nu_2 \mathbf{q}} = \nu q^2 \tag{2.12}$$

and are of no further interest here.

The equations for the remaining three variables $\hat{p}$, $\hat{u}^{(l)}$, and $\hat{s}$ read, explicitly,

$$(\gamma - 1)\, D_T q^2 \hat{p} + icq\hat{u}^{(l)} + (\gamma - 1)^{1/2}\, D_T q^2 \hat{s} = h\hat{p} \tag{2.13a}$$

$$icq\hat{p} + \Gamma_l q^2 \hat{u}^{(l)} = h\hat{u}^{(l)} \tag{2.13b}$$

$$(\gamma - 1)^{1/2}\, D_T q^2 \hat{p} + D_T q^2 \hat{s} = h\hat{s} \tag{2.13c}$$

This leads to the characteristic equation

$$h^3 - (\Gamma_l + \gamma D_T)\, q^2 h^2 + (c^2 q^2 + \gamma D_T \Gamma_l q^4) h - c^2 D_T q^4 = 0 \tag{2.14}$$

Although Eq. (2.14) can, in principle, be solved exactly, it is not necessary to do so, because it involves a small parameter in which the solution can be expanded. The parameter is

$$\varepsilon_1 = \Gamma q/c \ll 1 \tag{2.15}$$

where $\Gamma$ denotes any of the generalized diffusion coefficients. The condition (2.15), which imposes an upper bound on the wavevectors, is a basic requirement for the hydrodynamic equations to be applicable. In fact, (2.15) requires that the variables change smoothly in space, so that the dissipative terms (those involving transport coefficients) be much smaller than the streaming terms. Since the hydrodynamic equations are supposed to hold only up to first order in $\varepsilon_1$, it makes no sense to compute their solutions with a better accuracy.

Solving thus (2.14) by first-order perturbation theory, one obtains the eigenvalues

$$h_{H\mathbf{q}} = D_T q^2 \tag{2.16}$$

and

$$h_{\sigma\mathbf{q}} = i\sigma cq + \tfrac{1}{2}\Gamma_s q^2 \qquad (\sigma = \pm 1) \tag{2.17}$$

where $\Gamma_s = \Gamma_l + (\gamma - 1)\, D_T$ is the sound damping coefficient. The first solution, (2.16), corresponds to the heat mode, which will not be discussed here. The other two solutions, given by (2.17), are the eigenvalues of the sound modes.[3]

Using (2.17) in (2.13), one may also evaluate the eigenvectors of the sound modes. To zeroth order, one finds that $\hat{p}_{\sigma\mathbf{q}} = \sigma\hat{u}^{(l)}_{\sigma\mathbf{q}}$ and $\hat{s}_{\sigma\mathbf{q}} = 0$. There are, of course, first-order corrections to these results. However, in the following we will only be interested in the zeroth-order eigenvectors. Going back to (2.8), considering also the adjoint problem, and normalizing according to (2.4), one obtains finally

$$\begin{aligned}
p_{\sigma\mathbf{q}}(\mathbf{r}) &= \frac{1}{4\pi^{3/2}} \exp(i\mathbf{q}\cdot\mathbf{r}), & \bar{p}_{\sigma\mathbf{q}}(\mathbf{r}) &= \frac{1}{4\pi^{3/2}} \exp(-i\mathbf{q}\cdot\mathbf{r}) \\
\mathbf{u}_{\sigma\mathbf{q}}(\mathbf{r}) &= \frac{\sigma}{4\pi^{3/2}}\, \hat{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}), & \hat{\mathbf{u}}_{\sigma\mathbf{q}}(\mathbf{r}) &= \frac{\sigma}{4\pi^{3/2}}\, \hat{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{r}) \\
s_{\sigma\mathbf{q}}(\mathbf{r}) &= 0, & \bar{s}_{\sigma\mathbf{q}}(\mathbf{r}) &= 0
\end{aligned} \tag{2.18}$$

I conclude this section with two remarks that are important for the later discussion:

1. Knowing that $|h_{\sigma\mathbf{q}}| = O(cq)$, one may also calculate the first-order eigenvalues and the zeroth-order eigenvectors without using (2.14). To this end, consider the system (2.13). Since $|h| = O(cq)$, it follows from (2.13c) that $\hat{s} = O(\varepsilon_1)$; hence, the last term on the left-hand side of (2.13a) is of order $\varepsilon_1^2$ and can be neglected. Equations (2.13a) and (2.13b) then reduce to a system for $\hat{p}$ and $\hat{u}^{(l)}$ alone, namely

$$\begin{aligned}
(\gamma - 1)\, D_T q^2 \hat{p} + icq\hat{u}^{(l)} &= h\hat{p} \\
icq\hat{p} + \Gamma_l q^2 \hat{u}^{(l)} &= h\hat{u}^{(l)}
\end{aligned} \tag{2.19}$$

which leads to a quadratic equation that is considerably simpler than (2.14), namely

$$h^2 - \Gamma_s q^2 h + c^2 q^2 = 0 \tag{2.20}$$

The first-order solutions of (2.20) are, of course, in agreement with (2.17). Moreover, one finds from (2.19) that $\hat{p}_{\sigma\mathbf{q}} = \sigma\hat{u}^{(l)}_{\sigma\mathbf{q}}$ to zeroth order.

[3] I use $\sigma$ also as an index to label the two sound modes. Thus, (2.17) should be read as

$$h_{+\mathbf{q}} = +icq + \tfrac{1}{2}\Gamma_s q^2; \qquad h_{-\mathbf{q}} = -icq + \tfrac{1}{2}\Gamma_s q^2$$

2. Inserting (2.17) and (2.18) into (2.6), one may discuss the contribution of a sound mode $(\sigma\mathbf{q})$ to the general solution $\delta\mathbf{a}(\mathbf{r}, t)$. It is then observed that each mode $(\sigma\mathbf{q})$ gives rise to a term which behaves in space and time as

$$\mathbf{a}_{\sigma\mathbf{q}}(\mathbf{r}, t) \equiv \mathbf{a}_{\sigma\mathbf{q}}(\mathbf{r}) \exp(-h_{\sigma\mathbf{q}} t) \propto \exp[i\mathbf{q} \cdot (\mathbf{r} - \sigma\hat{\mathbf{q}}ct) - \tfrac{1}{2}\Gamma_s q^2 t]$$

thus representing a damped plane wave. The wavefront, located at $\mathbf{r}_{\sigma\mathbf{q}}(t) = \sigma\hat{\mathbf{q}}ct$, propagates uniformly with speed $c$ either in the direction of $\hat{\mathbf{q}}$ (if $\sigma = +$) or $-\hat{\mathbf{q}}$ (if $\sigma = -$). Furthermore, the wave is damped, the attenuation time being $\tau = 2/\Gamma_s q^2$. The attenuation length, i.e., the distance traveled by the wavefront during the time $\tau$, is therefore given by

$$l_{\mathbf{q}} = \frac{2c}{\Gamma_s q^2} \tag{2.21}$$

Notice that $l_{\mathbf{q}} = O(1/\varepsilon_1 q)$ [cf. (2.15)], implying that the attenuation length is of the same order as, or even larger than, the size of the system, $d$ say, for all wavevectors in the range $q \lesssim O(1/\varepsilon_1 d)$. Under these circumstances the sound waves hit the walls, and boundary effects might become important.[4]

## 3. EVALUATION OF THE SOUND MODES IN THE PRESENCE OF WALLS WITH ACOUSTIC BOUNDARY CONDITIONS

In this section I outline the calculation of the sound modes in the case that the fluid (still in equilibrium) is bounded. To be specific, let us consider a plane fluid layer of thickness $d$. Choose coordinates such that the walls coincide with the planes $z = +d/2$ and $z = -d/2$, respectively, and ignore boundaries in the $x$ and $y$ directions. Furthermore, let us impose so-called acoustic boundary conditions on the variables $\delta\mathbf{a}(\mathbf{r}, t)$ at the walls. These are given by

$$(\gamma - 1)^{1/2}\, \delta p + \delta s = 0 \tag{3.1a}$$

$$\delta u_x = 0, \qquad \delta u_y = 0 \qquad \left(z = \pm\frac{d}{2}\right) \tag{3.1b}$$

$$\delta u_z = \pm\beta \left(\delta p - \frac{\Gamma_l}{c} \frac{\partial \delta u_z}{\partial z}\right) \tag{3.1c}$$

[4] For diffusive modes, on the other hand, boundary effects are only relevant for very small wavevectors with $q \lesssim O(1/d)$.

where $\beta \geqslant 0$ is a dimensionless wall parameter, called the specific acoustic admittance.[(6)] In (3.1c), the plus sign applies at $z = +d/2$, while the minus sign holds at $z = -d/2$.

The physical meaning of (3.1) is as follows: (3.1a) requires that the (excess) temperature, to which the left-hand side of (3.1a) is proportional, vanishes at the walls (perfectly heat-conducting walls). Condition (3.1b) says that the two tangential components of $\delta\mathbf{u}$ vanish at the walls (perfect stick conditions). Finally, (3.1c) requires that the normal component of $\delta\mathbf{u}$ be proportional to the normal–normal component of the (excess) pressure tensor. The specific acoustic admittance is a measure of the elasticity of the wall: $\beta = 0$ corresponds to a perfectly rigid wall, while $\beta = \infty$ corresponds to a completely deformable, i.e., elastic wall. The fact that $\beta$ cannot be negative follows from thermodynamic reasons, since it describes dissipative processes.[(4)]

It should also be remarked that the boundary conditions (3.1) are a special case of the most general boundary conditions which one can naturally impose on the hydrodynamic operator.[(4)]

In order to solve the eigenvalue problem associated with Eqs. (2.1) and the boundary conditions (3.1), we use the fact that there is still translational invariance in the $x$, $y$ plane to make the ansatz

$$\mathbf{a}_K(\mathbf{r}) = \hat{\mathbf{a}}(z)\exp(i\mathbf{q}_{\|}\cdot\mathbf{r}_{\|}) \tag{3.2}$$

where $\mathbf{r}_{\|} = (x, y)$ and $\mathbf{q}_{\|} = (q_x, q_y)$ may be any (real) two-component "horizontal" wavevector. Furthermore, let us decompose the velocity $\mathbf{u}_K(\mathbf{r})$ according to

$$\mathbf{u}_K(\mathbf{r}) = \nabla\phi_K(\mathbf{r}) - (\mathbf{e}_z \times \nabla)\,\xi_K(\mathbf{r}) - \frac{1}{q_{\|}^2}(\mathbf{e}_z \times \nabla) \times \nabla v_K(\mathbf{r}) \tag{3.3}$$

where each of the scalar "potentials" $\phi_K$, $\xi_K$, and $v_K$ depends on $z$ and $\mathbf{r}_{\|}$ as indicated in (3.2).[5] From (3.3) it follows that

$$\begin{aligned} \nabla\cdot\mathbf{u}_K &= -\mathscr{D}\phi_K \\ \mathbf{e}_z\cdot(\nabla\times\mathbf{u}_K) &= \phi_K \\ \mathbf{e}_z\cdot[\nabla\times(\nabla\times\mathbf{u}_K)] &= -\mathscr{D}v_K \end{aligned} \tag{3.4}$$

with the operator

$$\mathscr{D} = -\nabla^2 = q_{\|}^2 - \frac{d^2}{dz^2} \tag{3.5}$$

[5] Notice that $\phi_K$ and $\xi_K$ have the "dimension" $q^{-1}$ because of the gradients in (3.3).

Notice from (3.4) that $\phi_K$ is the potential for the longitudinal velocity, whereas the two components of the transverse velocity can be derived from $\xi_K$ and $v_K$.

Inserting (3.2) and (3.3) into (2.3), with $\mathscr{H}$ given by (2.1), and using (3.4), one finds that the transverse part of the velocity, represented by $\hat{\xi}(z)$ and $\hat{v}(z)$, is decoupled from the other variables, as was the case for the unbounded fluid. It follows that

$$\hat{\xi}(z)=0, \qquad \hat{v}(z)=0 \tag{3.6}$$

for all but the viscous modes, which are not considered here. The remaining variables obey the coupled equations

$$(\gamma-1)\, D_T\mathscr{D}\hat{p}(z)-c\mathscr{D}\hat{\phi}(z)+(\gamma-1)^{1/2}\, D_T\mathscr{D}\hat{s}(z)=h\hat{p}(z) \tag{3.7a}$$

$$c\mathscr{D}\hat{p}(z)+\Gamma_l\mathscr{D}^2\hat{\phi}(z)=h\mathscr{D}\hat{\phi}(z) \tag{3.7b}$$

$$(\gamma-1)^{1/2}\, D_T\mathscr{D}\hat{p}(z)+D_T\mathscr{D}\hat{s}(z)=h\hat{s}(z) \tag{3.7c}$$

which are the analogues of (2.13) for the finite case. Moreover, requiring that the eigenvectors satisfy the boundary conditions (3.1), and using (3.2), (3.3), and (3.6), we obtain at the walls

$$(\gamma-1)^{1/2}\hat{p}+\hat{s}=0 \tag{3.8a}$$

$$\hat{\phi}=0 \qquad \left(z=\pm\frac{d}{2}\right) \tag{3.8b}$$

$$\frac{d\hat{\phi}}{dz}=\pm\beta\left(\hat{p}-\frac{\Gamma_l}{c}\frac{d^2\hat{\phi}}{dz^2}\right) \tag{3.8c}$$

Equations (3.7) and (3.8) define a well-posed one-dimensional eigenvalue problem, from which the sound modes as well as the heat modes can be obtained.

Before turning to this problem, I briefly discuss the adjoint problem. Upon deriving the adjoint eigenvalue equations and the adjoint boundary conditions from (2.1) and (3.1), one finds that the adjoint eigenvectors can be chosen according to

$$\bar{\mathbf{a}}_K(\mathbf{r})=\hat{\bar{\mathbf{a}}}(z)\exp(-i\mathbf{q}_{\|}\cdot\mathbf{r}_{\|}) \tag{3.9}$$

where

$$\hat{\bar{p}}(z)=\hat{p}(z), \qquad \hat{\bar{\mathbf{u}}}(z)=-\hat{\mathbf{u}}(z), \qquad \hat{\bar{s}}(z)=\hat{s}(z) \tag{3.10}$$

Hence, one does not have to solve the adjoint problem explicitly. Finally, one may insert (3.2) and (3.9), together with (3.3), (3.6), and (3.10), into (2.4) to derive the correct normalization of the eigenfunctions, which is found to be

$$\int_{-d/2}^{d/2} dz \,[\hat{p}^2(z) - \hat{\phi}(z)\,\mathscr{D}\hat{\phi}(z) + \hat{s}^2(z)] = \frac{1}{(2\pi)^2} \tag{3.11}$$

where use has also been made of the boundary condition (3.8b).

Let us now return to the eigenvalue problem (3.7), (3.8). One way to proceed consists in eliminating $\hat{p}(z)$ and $\hat{s}(z)$ from (3.7). This leads to an eigenvalue equation involving $\hat{\phi}(z)$ alone, namely

$$\{D_T(c^2 - h\gamma\Gamma_l)\,\mathscr{D}^2 - h[c^2 - h(\Gamma_l + \gamma D_T)]\,\mathscr{D} - h^3\}\,\mathscr{D}\hat{\phi}(z) = 0 \tag{3.12}$$

while $\hat{p}(z)$ and $\hat{s}(z)$ can be expressed in terms of the solutions of (3.12) as follows:

$$\hat{p}(z) = \frac{c}{h^2}\left[D_T\left(1 - h\frac{\gamma\Gamma_l}{c^2}\right)\mathscr{D} - h\left(1 - h\frac{\gamma D_T}{c^2}\right)\right]\mathscr{D}\hat{\phi}(z) \tag{3.13a}$$

$$\hat{s}(z) = \frac{c}{h^2(\gamma - 1)^{1/2}}\,D_T\left[\left(1 - h\frac{\gamma\Gamma_l}{c^2}\right)\mathscr{D} + h^2\frac{\gamma}{c^2}\right]\mathscr{D}\hat{\phi}(z) \tag{3.13b}$$

Equations (3.12) and (3.13) are, of course, fully equivalent to (3.7).

Since (3.12) has constant coefficients, an exponential ansatz

$$\hat{\phi}(z) \propto e^{\kappa z} \tag{3.14}$$

is appropriate. Inserting this into (3.12), one obtains a third-order algebraic equation for $q^2 = q_{\|}^2 - \kappa^2$ [which is identical to (2.14), except for an overall factor $q^2$]. Solving this equation for $q^2$ at fixed $h$, one obtains at leading order in $|h|\,\Gamma/c^2$ [which plays the role of $\varepsilon_1$, as will be argued in (4.1) below] the three (possibly complex) solutions

$$q_1^2(h) = 0, \qquad q_2^2(h) = -\frac{h^2}{c^2} + \cdots, \qquad q_3^2(h) = \frac{h}{D_T} + \cdots \tag{3.15}$$

where the dots indicate higher order corrections. Equations (3.15) lead to six characteristic wavenumbers, given by $\kappa_{j\pm}(h) = \pm[q_{\|}^2 - q_j^2(h)]^{1/2}$ ($j = 1, 2, 3$). Hence, the general solution $\hat{\phi}(z)$ of (3.12) is a linear combination of the six exponential function $\exp \kappa_{j\pm} z$, involving six unknown constants. Upon computing now $\hat{p}(z)$ and $\hat{s}(z)$ according to (3.13) and applying the boundary conditions (3.8) (three on each wall), one obtains a linear and homogeneous system of six equations for the six constants. The condition that this system have nontrivial solutions finally leads to the characteristic equation from which the eigenvalue spectrum is obtained.

Although the method described above is conceptually simple and correct, I shall not proceed in this way here, since the actual calculation gets rather involved. This is mainly due to the fact that the full information about the heat modes (in which we are ultimately not interested) has to be carried along. I instead prefer to use the small parameter $\varepsilon_1$ [cf. (2.15)] at an early stage of the calculation. In the next section I will outline how this can be achieved by applying a so-called *singular perturbation theory*.[7] Knowing how this method works is even more important in the nonequilibrium case, where the starting equations are far more complicated.

## 4. SINGULAR PERTURBATION THEORY

In this section I outline how the sound modes in the bounded system are obtained by using a singular perturbation theory. In short, the basic idea is as follows: It is implied by the results found in (3.15) that the spatial variation of the eigenfunctions takes place on three different length scales, which are determined by the characteristic wavelengths $|\kappa_j(h)|^{-1} = |q_{\|}^2 - q_j^2(h)|^{-1/2}$ $(j = 1, 2, 3)$. I shall argue that the scale $|\kappa_3|^{-1}$ is much shorter than $|\kappa_1|^{-1}$ and $|\kappa_2|^{-1}$, and that the short-scale changes occur only near the walls. This will make it possible to derive two sets of equations, one for the bulk region away from the walls, and another for the boundary layers. Both sets are considerably simpler than (3.7) [resp. (3.12), (3.13)] and will be solved separately. Finally, the two solutions will be matched analytically.

First notice from (3.15) that the branch $q_2^2(h)$ coincides with the leading-order "dispersion relation" for the sound modes in the infinite system [$h^2 = -c^2q^2$; cf. (2.17)]. The contributions to the eigenfunctions arising from the other two branches, $q_1^2$ and $q_3^2$, must therefore vanish, or at least be small, in the limit $d \to \infty$ at fixed $z$. Since the branch $q_2^2(h)$ leads to the bulk wavevectors of the sound modes, one can identify the $q$ appearing in (2.15) with $|q_2| = |h|/c$, to obtain

$$\varepsilon_1 = \frac{|h|\,\Gamma}{c^2} \ll 1 \tag{4.1}$$

as a small parameter. Recall that this condition guarantees that the solutions lie in the hydrodynamic regime.

Assuming that $\operatorname{Re} q_2^2 \geqslant q_{\|}^2$ (Re denotes the real part),[6] one may easily show with the aid of (3.15) and (4.1) that $|\kappa_1| = q_{\|} \leqslant |h|/c$ and $|\kappa_2| \leqslant |h|/c$, while $|\kappa_3| = O(\varepsilon_1^{-1/2}\,|h|/c)$. Thus, the scale $|\kappa_3|^{-1}$ is a factor $\varepsilon_1^{1/2}$ smaller than the other two scales. Moreover, it is obvious that changes on the scale $|\kappa_3|^{-1}$ can only occur near the walls, for, otherwise, one would be unable to recover the infinite-system results (which do not involve that scale) in the limit $d \to \infty$ at any fixed $z$.

From the above considerations it follows that it is possible to discuss the eigenvalue equations in the bulk region and in the boundary layers separately. The bulk region is the regime sufficiently far away from the walls such that all terms changing on the scale $|\kappa_3|^{-1}$ have essentially dropped to zero. Hence, all $z$ satisfying the condition $-d/2 + \varepsilon_1^{1/2} c/|h| \ll z \ll d/2 - \varepsilon_1^{1/2} c/|h|$ belong to the bulk region. The boundary layers, on the other hand, are the regions close to the walls where changes occur predominantly on the scale $|\kappa_3|^{-1}$, whereas changes on the scales $|\kappa_1|^{-1}$ and $|\kappa_2|^{-1}$ can be neglected. Hence, the boundary layers consist of all $z$ satisfying the conditions $-d/2 \leqslant z \ll -d/2 + c/|h|$ or $d/2 - c/|h| \ll z \leqslant d/2$, respectively. Notice that the bulk region and the boundary layers are partially overlapping. This will make it possible to match the solutions to be obtained in each regime.

I first discuss the bulk region, where spatial variations occur only on the scales $|\kappa_1|^{-1}$, $|\kappa_2|^{-1} \geqslant c/|h|$, implying that $\mathscr{D}$ is of the order $|h|^2/c^2$. For this reason one can apply to (3.7) the same arguments that were used before in going from (2.13) to (2.19). Namely, from (3.7c) one finds that $\hat{s}(z)$ is of the order $\varepsilon_1$. Hence, the last term on the left-hand side of (3.7a) is of the order $\varepsilon_1^2$ and can be neglected. Equations (3.7a) and (3.7b) then become a closed system for $\hat{p}(z)$ and $\hat{\phi}(z)$ alone, from which one can eliminate $\hat{p}(z)$ to obtain

$$[(c^2 - h\Gamma_s)\,\mathscr{D} + h^2]\,\mathscr{D}\hat{\phi}(z) = 0 \tag{4.2}$$

This equation is the analogue of (2.20) and holds consistently up to first order in $\varepsilon_1$. For completeness I also quote the zeroth-order results for $\hat{p}(z)$ and $\hat{s}(z)$ in the bulk region:

$$\hat{p}_b(z) = -\frac{c}{h}\,\mathscr{D}\hat{\phi}(z) \tag{4.3a}$$

$$\hat{s}_b(z) = 0 \tag{4.3b}$$

[6] This is equivalent to assuming that the quantity $Q_z^2(h)$ defined by $q_2^2(h) = q_{\|}^2 + Q_z^2(h)$ has a nonnegative real part. In the infinite system this is always the case, since $Q_z$ is here the $z$ component of the real wavevector. Remark, however, that the condition $\operatorname{Re} Q_z^2(h) \geqslant 0$ does not hold for certain "interfacial modes,"(8) which are not considered here.

Equation (4.2) is easily solved, the general solution being

$$\hat{\phi}(z) = \hat{\phi}_b(z) \equiv Ae^{-q_{\|}z} + Be^{q_{\|}z} + Ce^{\Lambda(h)z} + De^{-\Lambda(h)z} \tag{4.4}$$

where the label $b$ indicates "bulk,"

$$\Lambda(h) = \left(q_{\|}^2 + \frac{h^2}{c^2 - h\Gamma_s}\right)^{1/2} \qquad \left(-\frac{\pi}{2} < \arg \Lambda(h) \leqslant \frac{\pi}{2}\right) \tag{4.5}$$

and $A$, $B$, $C$, and $D$ are arbitrary constants. In (4.5) the main branch of the square root is understood, as indicated in brackets.

Next let us turn to the boundary layers, where variations on the scale $|\kappa_3|^{-1}$ occur, so that $d^2/dz^2$ is of the order $\varepsilon_1^{-1}\,|h|^2/c^2 \gg q_{\|}^2$. Since the thickness of these layers is small compared to $|\kappa_1|^{-1}$ and $|\kappa_2|^{-1}$, we need only keep the leading order terms in $\varepsilon_1$. Thus, we derive from (3.12) the equation

$$\left(D_T \frac{d^2}{dz^2} + h\right) \frac{d^4}{dz^4} \hat{\phi}(z) = 0 \tag{4.6}$$

and from (3.13)

$$\hat{p}(z) = \frac{c}{h^2}\left(D_T \frac{d^2}{dz^2} + h\right) \frac{d^2}{dz^2} \hat{\phi}(z) \tag{4.7a}$$

$$\hat{s}(z) = \frac{c}{h^2(\gamma - 1)^{1/2}} D_T \frac{d^4}{dz^4} \hat{\phi}(z) \tag{4.7b}$$

Equation (4.6) is now solved in each layer separately. In the layer close to $z = d/2$, the general solution of (4.6) can be written as a linear combination involving six terms, two of which are proportional to $\exp[\pm\lambda(h)(z - d/2)]$, respectively, where

$$\lambda(h) = \left(-\frac{h}{D_T}\right)^{1/2} \qquad \left(-\frac{\pi}{2} < \arg \lambda(h) \leqslant \frac{\pi}{2}\right) \tag{4.8}$$

while the other four terms yield a third-order polynomial in $(z - d/2)$. The term $\exp[-\lambda(h)(z - d/2)]$ must be discarded, since it increases exponentially away from the wall. The remainder contains five unknown constants, three of which can be determined by applying the boundary conditions (3.8) at $z = d/2$. Thus, one finds

$$\hat{\phi}(z) = \hat{\phi}^{(+)}(z) \equiv E_+ \left\{ \exp\left[\lambda(h)\left(z-\frac{d}{2}\right)\right] - 1 - \left[\lambda(h) + \frac{c}{(\gamma-1)\,D_T}\bar{\beta}(h)\right]\left(z-\frac{d}{2}\right) - \frac{h}{2(\gamma-1)\,D_T}\left(z-\frac{d}{2}\right)^2 \right\} - \frac{1}{6}F_+\lambda^3(h)\left(z-\frac{d}{2}\right)^3 \tag{4.9}$$

where $E_+$, $F_+$ are arbitrary constants and

$$\bar{\beta}(h) = \left(1 - h\frac{\gamma\Gamma_l}{c^2}\right)\beta \tag{4.10}$$

I also quote the results for $\hat{p}(z)$ and $\hat{s}(z)$ near the wall, which are

$$\hat{p}^{(+)}(z) = -\frac{c}{(\gamma-1)\,D_T}\left[E_+ - F_+(\gamma-1)\,\lambda(h)\left(z-\frac{d}{2}\right)\right] \tag{4.11a}$$

$$\hat{s}^{(+)}(z) = \frac{c}{(\gamma-1)^{1/2}\,D_T}E_+\exp\left[\lambda(h)\left(z-\frac{d}{2}\right)\right] \tag{4.11b}$$

Notice that $\hat{p}^{(+)}(z)$ does not involve an exponentially decaying part.

In the other boundary layer, located near $z = -d/2$, the solution is obtained in the same way. Here one finds

$$\hat{\phi}(z) = \hat{\phi}^{(-)}(z) \equiv E_- \left\{ \exp\left[-\lambda(h)\left(z+\frac{d}{2}\right)\right] - 1 + \left[\lambda(h) + \frac{c}{(\gamma-1)\,D_T}\bar{\beta}(h)\right]\left(z+\frac{d}{2}\right) - \frac{h}{2(\gamma-1)\,D_T}\left(z+\frac{d}{2}\right)^2 \right\} + \frac{1}{6}F_-\lambda^3(h)\left(z+\frac{d}{2}\right)^3 \tag{4.12}$$

where $E_-$, $F_-$ are arbitrary constants, and expressions for $\hat{p}(z)$ and $\hat{s}(z)$ that are similar to (4.11).

In the final step of the calculation, the general solutions, found so far in the bulk region and the boundary layers separately, have to be matched in the overlap region

$$-\frac{d}{2} + \varepsilon_1^{1/2}\frac{c}{|h|} \ll z \ll -\frac{d}{2} + \frac{c}{|h|}, \qquad \frac{d}{2} - \frac{c}{|h|} \ll z \ll \frac{d}{2} - \varepsilon_1^{1/2}\frac{c}{|h|}$$

respectively. I outline the method only for the overlap region near the wall at $z = d/2$, since the other overlap region is treated in the same way.

For $z$ values in the overlap region near the wall at $z=d/2$, the formulas (4.4) and (4.9) both apply. Moreover, the $z$ values in this range are sufficiently *far* away from the wall for the exponential term in $\hat{\phi}^{(+)}(z)$ to be negligible. Yet, they are *close* enough to be wall so that only a few terms in a Taylor expansion of $\hat{\phi}_b(z)$ about $z=d/2$ need to be taken into account. Expanding, therefore, $\hat{\phi}_b(z)$ up to third order in $(z-d/2)$, and comparing the result with the third-order polynomial appearing in $\hat{\phi}^{(+)}(z)$, one obtains four relations between $A$, $B$, $C$, and $D$ and $E_+$, $F_+$, which I do not write down here.

For the other overlap region, near the wall at $z=-d/2$, the procedure is the same, resulting in four relations between $A$, $B$, $C$, and $D$ and $E_-$, $F_-$. In total, thus, one obtains eight linear and homogeneous equations for the eight unknown constants in (4.4), (4.9), and (4.12).

The eigenvalue spectrum is obtained by solving the characteristic equation of this $8\times 8$ system. Moreover, for each eigenvalue one finds a set of coefficients $A$, $B$, $C$, $D$, $E_+$, $F_+$, $E_-$, and $F_-$, which is uniquely determined up to a normalization constant. Inserting these results into the above expressions for $\hat{\phi}(z)$, $\hat{p}(z)$, and $\hat{s}(z)$, one finds all the eigenfunctions, which are, finally, normalized according to (3.11).

In the remainder of this section I only quote the characteristic equation and briefly outline its solution. Complete expressions for the eigenvalues and eigenfunctions may be obtained from Section 7 by putting all parameters equal to constants, and a discussion of these results is given in Section 8.

Since the system is invariant under a reflection at the plane $z=0$, it follows that the modes have either odd or even parity. For each class one finds a separate characteristic equation, which reads

$$\exp \Lambda(h)d = -G(h) \tag{4.13a}$$

for the odd modes, and

$$\exp \Lambda(h)d = G(h) \tag{4.13b}$$

for the even modes. Here

$$G(h)=\frac{h[\Lambda(h)+q_{\|}]+c\beta(h)[\Lambda^2(h)-q_{\|}^2]}{h[\Lambda(h)-q_{\|}]-c\beta(h)[\Lambda^2(h)-q_{\|}^2]} \tag{4.14}$$

and $\Lambda(h)$ is given by (4.5).

Actually the calculation yields that the $q_{\|}$ appearing in the terms $[\Lambda(h)+q_{\|}]$ and $[\Lambda(h)-q_{\|}]$ of (4.14) should be replaced by $q_{\|}\coth q_{\|}d/2$ for the odd modes and by $q_{\|}\tanh q_{\|}d/2$ for the even modes. I have omitted the factors coth $q_{\|}d/2$ and tanh $q_{\|}d/2$ here, since they are only significantly different from 1 if $q_{\|}d \leqslant 1$. In this case, however, $q_{\|}$ coth $q_{\|}d/2$ and $q_{\|}$ tanh $q_{\|}d/2$ are of the order $1/d$, which is negligible compared to $|\Lambda(h)|$ for the modes in which we are ultimately interested [cf. (4.15)–(4.17)].

From (4.13) one finds first that

$$\Lambda(h) = iq_{zn} + \frac{1}{d}\log G(h) \qquad (-\pi < \arg \log G(h) \leqslant \pi) \tag{4.15}$$

where $q_{zn}$ is a discrete wavenumber, defined by

$$q_{zn} = n\pi/d \tag{4.16}$$

and $n$ may be any positive integer. Furthermore, the main branch of the logarithm is understood. Note that negative integers $n$ have to be excluded in (4.15) due to restriction on arg $\Lambda(h)$ [cf. (4.5)]. For each $n$, Eq. (4.15) yields two eigenvalues, denoted by $h_{\sigma \mathbf{q}_n}(\sigma = \pm)$, where $\mathbf{q}_n = (\mathbf{q}_{\|}, q_{zn})$.

To obtain these explicitly, I restrict consideration to modes for which the wavelengths $q_{zn}^{-1}$ are small compared to $d$. Then one has, in addition to $\varepsilon_1$, a second small parameter, namely

$$\varepsilon_2 = \frac{1}{n\pi} \ll 1 \tag{4.17}$$

For these modes the second term on the right-hand side of (4.15) is small, and the solutions $h_{\sigma \mathbf{q}_n}$ are straightforwardly obtained by applying a first-order perturbation theory in $\varepsilon_1$ and $\varepsilon_2$.

## 5. EVALUATION OF THE SOUND MODES IN THE PRESENCE OF A STEADY HEAT FLUX

Let us now consider a more complicated situation, in which the fluid is no longer in equilibrium. The geometry is chosen as in the preceding sections; however, let us now suppose that the two walls, located at $z = \pm d/2$, are maintained at different temperatures, $T^{(\pm)}$, respectively. As a result, a steady heat flux in the $z$ direction is established, giving rise to a one-dimensional temperature profile $T(z)$, which is determined by the equation of heat conduction

$$\frac{d}{dz}\kappa(T(z))\frac{dT}{dz} = 0$$

where $\kappa(T)$ is the thermal conductivity of the fluid.

The temperature profile introduces a new (local) macroscopic length $L_\nabla$, given by

$$L_\nabla(z) = \left| \frac{d \ln T}{dz} \right|^{-1} \tag{5.1}$$

which may compete with $d$. $L_\nabla$ is infinite in equilibrium and gets smaller the further away the system is from equilibrium. I exclude, however, the extreme case $L_\nabla \ll d$. Hence, the temperature is still of the same order of magnitude throughout the system. Moreover, since all parameters $P = c, \gamma, \Gamma, \ldots$ appearing in the hydrodynamic equations depend on temperature via local equations of state, it follows that they become functions of $z$, too. Let us assume that the equations of state are such that $\partial \ln P / \partial \ln T$ is of the order 1 for all $P$ in the range of temperatures being considered.[7] This implies that none of the parameters change on a scale much smaller than $L_\nabla$.

To define the eigenvalue problem for the sound modes in a proper way, one starts from the hydrodynamic equations, linearized about the steady state, and the boundary conditions. The linearized equations read

$$\begin{aligned}
\frac{\partial}{\partial t} \delta p &= \nabla \cdot (\gamma - 1)\, D_T \nabla\, \delta p - c \nabla \cdot \delta \mathbf{u} - \frac{1}{2} c\alpha \frac{dT}{dz} \delta u_z + \nabla \cdot (\gamma - 1)^{1/2} D_T \nabla\, \delta s \\
\frac{\partial}{\partial t} \delta \mathbf{u} &= -\nabla c\, \delta p + \frac{1}{2} c\alpha \frac{dT}{dz} \mathbf{e}_z\, \delta p + \nabla(\Gamma_l - \nu) \nabla \cdot \delta \mathbf{u} + \nabla \cdot \nu \nabla\, \delta \mathbf{u} \\
\frac{\partial}{\partial t} \delta s &= \nabla \cdot (\gamma - 1)^{1/2} D_T \nabla\, \delta p - \frac{c\alpha}{(\gamma - 1)^{1/2}} \frac{dT}{dz} \delta u_z + \nabla \cdot D_T \nabla\, \delta s
\end{aligned} \tag{5.2}$$

where the position dependence of the parameters has not been indicated explicitly. Moreover, $\mathbf{e}_z$ is the unit vector in the $z$ direction and $\alpha$ is the thermal expansion coefficient. Impose again acoustic boundary conditions, given by

$$\begin{aligned}
&(\gamma - 1)^{1/2}\, \delta p + \delta s = 0 \\
&\delta u_x = 0, \qquad \delta u_y = 0 \qquad \left( z = \pm \frac{d}{2} \right) \\
&\delta u_x = \pm \beta^{(\pm)} \left( \delta p - \frac{\Gamma_l}{c} \frac{\partial \delta u_z}{\delta z} \right)
\end{aligned} \tag{5.3}$$

Notice that the walls at $z = \pm d/2$ are allowed to have different acoustic admittances, $\beta^{(\pm)}$, respectively, to include the possibility that $\beta$ is also a

[7] Steady states near phase transitions are thus excluded from the present discussion.

function of temperature, or that the walls are of a different nature. Furthermore, it is understood in (5.3) that the limits as $z \to \pm d/2$ of the fluid parameters are to be used.

It should be remarked that among the dissipative terms only those proportional to $\nabla \cdot \Gamma \nabla$ have been kept in (5.2), while corrections of the form $d\Gamma/dz\, \partial_z$ and $d^2\Gamma/dz^2$ have been dropped. Moreover, a term proportional to $d\Gamma/dz$ has been omitted in the last member of (5.3). All these terms are negligible in the approximation to be made, as will be indicated below.

In order to solve the eigenvalue problem associated with (5.2) and (5.3), one makes again the ansatz (3.2) (since there is still translational invariance in the $x$, $y$ plane), and proceeds as in Section 3 to derive a system of one-dimensional equations and boundary conditions, similar to (3.7) and (3.8). Since these equations are somewhat lengthy, I shall not quote them here. Instead, I restrict consideration to describing briefly their structure and giving an outline of how they are simplified.

One finds again that $\hat{\xi}$ is decoupled from the other equations, and thus zero for the sound modes. In contrast to the equilibrium case, however, one finds that the other transverse potential $\hat{v}$ is coupled to $(\hat{p}, \hat{\phi}, \hat{s})$ via the temperature gradient, which results in four equations for $(\hat{p}, \hat{\phi}, \hat{v}, \hat{s})$. Moreover, since $\hat{v}$ cannot be put equal to zero, it has to be kept in the boundary conditions, too.

Due to the fact that the parameters involved in these equations do not change by orders of magnitude throughout the system, one may generalize (4.1) to obtain the local condition

$$\varepsilon_1 = \frac{|h|\, \Gamma(z)}{c^2(z)} \ll 1 \tag{5.4}$$

To keep the problem analytically tractable, assume in addition that

$$\varepsilon_3 = \frac{c(z)}{|h|\, L_\nabla(z)} \ll 1 \tag{5.5}$$

This new condition imposes a lower bound on the eigenvalue spectrum, whereas (5.4) yields an upper bound. The main purpose of (5.5) is to keep the effect caused by the terms proportional to $dT/dz$ in (5.2) small compared to that of the leading streaming terms.

In view of (5.4) and (5.5), one may argue that the eigenfunctions change essentially on the same length scales as in equilibrium. These follow from (3.15), all parameters being replaced by their local values.[8] In par-

[8] Actually, the variable $\hat{v}$ adds a new branch to (3.15), given by $q_4^2 = h/v + \cdots$. However, $q_4^2$ is of the same order as $q_3^2$.

ticular, the branch $q_2^2(h; z) = -h^2/c^2(z)$ leads to the *local* bulk wavevectors, so that conditions (5.4) and (5.5) may also be interpreted as restrictions on the local wavelength $|q_2(z)|^{-1} = c(z)/|h|$. Moreover, one can again distinguish between a bulk region and two boundary layers and therefore apply a singular perturbation theory, as outlined in the last section.

The next task is thus to derive separate equations for the bulk region and the boundary layers. In the bulk region, where $\mathscr{D}$ is of the order $|h|^2/c^2$, one can again derive a closed system for $\hat{p}$ and $\hat{\phi}$ alone, by arguing that all couplings involving $\hat{v}$ and $\hat{s}$ are of the order $\varepsilon_1^2$ and that all corrections to the dissipative terms not explicitly quoted in (5.2) are of the order $\varepsilon_1 \varepsilon_3$. Eliminating $\hat{p}$ from this system, one obtains, consistent up to the first order,

$$[\mathscr{D}(c^2 - h\Gamma_s) + h^2]\, \mathscr{D}\hat{\phi}(z) = 0 \tag{5.6}$$

This equation is the same as (4.2), *except* that $c$ and $\Gamma_s$ are now given functions of $z$.

In the boundary layers, where $d^2/dz^2$ is of the order $\varepsilon_1^{-1}\, |h|^2/c^2 \gg q_{\|}^2$, one need only keep the leading terms, as was agreed in Section 4. To this order, $\hat{v}$ gets decoupled from $(\hat{p}, \hat{\phi}, \hat{s})$, and the corrections to the dissipative terms can be neglected, too. The equations are thus essentially the same as in equilibrium, so that one can use the general solutions $\phi^{(+)}(z)$ and $\phi^{(-)}(z)$ given in (4.9) and (4.12), with the minor modification that $\beta$ and the fluid parameters have to be replaced by their local values at the walls $z = +d/2$ and $z = -d/2$, respectively. The fact that the boundary layer solutions are essentially as in equilibrium is not very surprising after all, since the thickness of these layers is much smaller than $L_\nabla$.

Since we already know the solution in the boundary layers, there remains only the bulk equation (5.6) to be solved. This is achieved by using the *WKB method*,[9] as will be discussed next.

## 6. WKB METHOD

In this section I discuss how the general solution of the bulk equation (5.6) is obtained by means of the WKB method and, subsequently, how the characteristic equation for the sound modes in the presence of a heat flux is solved. The complete solution, i.e., the eigenvalues and eigenvectors, will then be presented in the next section.

To solve Eq. (5.6), first put

$$\hat{\psi}(z) = (c^2 - h\Gamma_s)\, \mathscr{D}\hat{\phi}(z) \tag{6.1}$$

and make the WKB ansatz

$$\hat{\psi}(z) = \exp \Psi(z) \tag{6.2}$$

which defines a new function $\Psi(z)$. Inserting (6.1) and (6.2) into (5.6), we obtain the following equation for $\Psi(z)$:

$$\left(\frac{d\Psi}{dz}\right)^2 + \frac{d^2\Psi}{dz^2} = q_{\|}^2 + \frac{h^2}{c^2(z) - h\Gamma_s(z)} \tag{6.3}$$

In the equilibrium case, where $c$ and $\Gamma_s$ are constants, two solutions of (6.3) are given by $\Psi(z) = \pm\Lambda(h)z$, with $\Lambda(h)$ as defined in (4.5). To solve (6.3) also for nonconstant coefficients, assume that $d^2\Psi/dz^2$ (which is strictly zero in equilibrium) is small compared to $(d\Psi/dz)^2$. Applying first-order perturbation theory, one then obtains the following two solutions:

$$\Psi(z) = \pm \int_{-d/2}^{z} dz'\, \Lambda(h; z') - \tfrac{1}{2} \log \Lambda(h; z) \tag{6.4}$$

where

$$\Lambda(h; z) = \left(q_{\|}^2 + \frac{h^2}{c^2(z) - h\Gamma_s(z)}\right)^{1/2} \qquad \left(-\frac{\pi}{2} < \arg \Lambda(h; z) \leqslant \frac{\pi}{2}\right) \tag{6.5}$$

For the perturbative solutions given in (6.4) to be consistent, we have to require that

$$\varepsilon_3' = \left|\frac{1}{\Lambda(h; z)} \frac{d \log \Lambda(h; z)}{dz}\right| \ll 1 \tag{6.6}$$

This is actually a stronger condition than (5.5), as I shall discuss in Section 7.

To construct now the general solution of (5.6), notice from (6.1)–(6.3) that $\mathscr{D}\hat{\psi} = -h^2\mathscr{D}\hat{\phi}$. Hence, two solutions $\hat{\phi}_1(z)$ and $\hat{\phi}_2(z)$ can be chosen to be proportional to the two functions obtained by inserting (6.4) into (6.2). These are

$$\hat{\phi}_1(z) = \frac{1}{[\Lambda(h; z)]^{1/2}} \exp \int_{-d/2}^{z} dz'\, \Lambda(h; z') \tag{6.7a}$$

$$\hat{\phi}_2(z) = \frac{1}{[\Lambda(h; z)]^{1/2}} \exp - \int_{-d/2}^{z} dz'\, \Lambda(h; z') \tag{6.7b}$$

Two more solutions of (5.6) are obviously given by $\hat{\phi}(z) = \exp(\pm q_{\|} z)$. The general solution of (5.6) can therefore be written in the form

$$\hat{\phi}(z) = \hat{\phi}_b(z) \equiv Ae^{q_{\|}z} + Be^{-q_{\|}z} + C\hat{\phi}_1(z) + D\hat{\phi}_2(z) \tag{6.8}$$

where $A$, $B$, $C$, and $D$ are arbitrary constants.

To determine the constants, one has to match the bulk solution (6.8) to the boundary layer solutions given by (4.9) and (4.12) (with the small modifications indicated in the last section). Proceeding as in Section 4, one then finds a linear and homogeneous system of eight equations for the eight unknown constants $A$, $B$, $C$, $D$, $E_+$, $F_+$, $E_-$, and $F_-$. This system leads to the characteristic equation

$$\exp 2 \int_{-d/2}^{d/2} dz\, \Lambda(h; z) = G^{(+)}(h)\, G^{(-)}(h) \tag{6.9}$$

where $G(h)$ is given by (4.14),[9] the upper labels $(+)$ and $(-)$ indicating that the parameter values at the wall $z = +d/2$ and $z = -d/2$, respectively, are to be used.

Equation (6.9) is the generalization of (4.13) in the presence of a heat flux. While in the equilibrium case we obtained separate characteristic equations for the odd and for the even modes, we have here only one characteristic equation for all modes, since reflection symmetry is lost.

I next discuss how Eq. (6.9) is solved. First one finds from (6.9) that

$$\int_{-d/2}^{d/2} dz\, \Lambda(h; z) = in\pi + \tfrac{1}{2}[\log G^{(+)}(h) + \log G^{(-)}(h)] \tag{6.10}$$

where $n$ may be any positive integer, and the main branches of the logarithms are implied. Negative integers $n$ are again to be excluded due to the restriction on $\arg \Lambda(h; z)$ [cf. (6.5)].

To solve (6.10) for each $n$, I restrict consideration, as in Section 4, to modes for which $\varepsilon_2 = 1/n\pi \ll 1$ and use perturbation theory in $\varepsilon_1$ and $\varepsilon_2$. I first discuss the zeroth order, in which the second term on the right-hand side of (6.10) and the sound damping coefficient in (6.5) can be neglected. This leads to the equation

$$\int_{-d/2}^{d/2} dz\, q_z(z) = n\pi \tag{6.11}$$

where $q_z(z)$ is given by

$$iq_z(z) = \left( q_{\|}^2 + \frac{h^2}{c^2(z)} \right)^{1/2} \tag{6.12}$$

Equation (6.11) is a generalization of (4.16) that determines a local, positive wavenumber $q_{nz}(z)$ for each $n$, which can be obtained in the following way.

[9] I omit again factors like $\cosh q_{\|} d/2$ and $\tanh q_{\|} d/2$ for reasons explained in Section 4.

First it follows from (6.12) that

$$c^2(z)[q_{\|}^2+q_z^2(z)] = -h^2 = c^2[q_{\|}^2+q_z^2(z_0)]$$

for any "reference point" $z_0$, since $h^2$ is independent of position. This relation allows us to derive the whole function $q_z(z)$ from its value in the reference point $z_0$:

$$q_z(z)=q_z(z_0)\frac{c(z_0)}{c(z)}\left[1+\frac{c^2(z_0)-c^2(z)}{c^2(z_0)}\frac{q_{\|}^2}{q_z^2(z_0)}\right]^{1/2} \tag{6.13}$$

Upon inserting (6.13) into (6.11), one can, in principle, compute the integral over $z$ to obtain an equation for $q_z(z_0)$, whose solution is denoted by $q_{nz}(z_0)$. This value may, in turn, be inserted into (6.13) to yield $q_{nz}(z)$ for all $z$.

As will be discussed in Section 8, the position-dependent wavenumbers $q_{nz}(z)$ describe the bending of sound due to the inhomogeneous sound velocity $c(z)$. It should also be emphasized that the existence and uniqueness of the $q_{nz}(z)$ at every point are guaranteed (for the modes considered here) due to assumption (6.6), which excludes the possibility that $\Lambda = iq_z$ can locally become zero.[10]

After the $q_{nz}(z)$ have been obtained in the manner described above, one can go back to (6.12) and solve for the zeroth-order eigenvalues. Introducing the local wavevectors

$$\mathbf{q}_n(z)=(\mathbf{q}_{\|}, q_{nz}(z)) \tag{6.14}$$

one then obtains

$$h_{\sigma\mathbf{q}_n}^{(0)} = i\sigma c(z)\, q_n(z) \tag{6.15}$$

where $q_n(z)=|\mathbf{q}_n(z)|$, and the label (0) indicates the zeroth order. Recall that the right-hand side of (6.15) does not depend on $z$, although $c(z)$ and $q_n(z)$ separately do.

Next let us evaluate the first-order corrections. Putting $h_{\sigma\mathbf{q}_n} = h_{\sigma\mathbf{q}_n}^{(0)} + h_{\sigma\mathbf{q}_n}^{(1)}$, where $h_{\sigma\mathbf{q}_n}^{(1)}$ denotes the first-order correction, we obtain by expanding (6.5) and using (6.15)

$$\Lambda(h_{\sigma\mathbf{q}_n}; z) = iQ_{z,\sigma\mathbf{q}_n}(z) \tag{6.16}$$

where $Q_{z,\sigma\mathbf{q}_n}(z)$ is a local, complex wavenumber given by

$$Q_{z,\sigma\mathbf{q}}(z) = q_{nz}(z) + i\sigma\left[\frac{1}{l_{\mathbf{q}_n}(z)} - \frac{h_{\sigma\mathbf{q}_n}^{(1)}}{c(z)}\right]\frac{1}{\hat{q}_{nz}(z)} \tag{6.17}$$

[10] In that case *total* sound bending would occur,[5] and the methods described above would have to be modified.

Here $l_{\mathbf{q}_n}(z)$ is the local attenuation length, defined by

$$l_{\mathbf{q}_n}(z)=\frac{2c(z)}{\Gamma_s(z)\, q_n^2(z)} \tag{6.18}$$

[cf. (2.21)] and $\hat{q}_{nz}=q_{nz}/q_n$. Moreover, I introduce the mode-dependent wall-absorption coefficients $b_{\sigma\mathbf{q}_n}=\frac{1}{2}\sigma\log G(h_{\mathbf{q}_n})$ with $G(h)$ given by (4.14). Using the zeroth-order results (6.15) and $\Lambda(h^{(0)}_{\sigma\mathbf{q}_n})=iq_{nz}$, one finds

$$b_{\sigma\mathbf{q}_n}=\frac{\sigma}{2}\log\frac{q_{nz}-iq_{\|}+\sigma\bar{\beta}_{\sigma\mathbf{q}_n}q_n}{q_{nz}+iq_{\|}-\sigma\bar{\beta}_{\sigma\mathbf{q}_n}q_n} \tag{6.19}$$

where $\bar{\beta}_{\sigma\mathbf{q}_n}=\bar{\beta}(h^{(0)}_{\sigma\mathbf{q}_n})$ is found from (4.10) to be given by

$$\bar{\beta}_{\sigma\mathbf{q}_n}=\left(1-i\sigma\frac{\gamma\Gamma_l q_n}{c}\right)\beta \tag{6.20}$$

Using (6.16) and (6.19), one can write Eq. (6.10) in the form

$$\int_{-d/2}^{d/2} dz\, Q_{z,\sigma\mathbf{q}_n}(z)=n\pi-i\sigma[b^{(+)}_{\sigma\mathbf{q}_n}+b^{(-)}_{\sigma\mathbf{q}_n}] \tag{6.21}$$

Upon inserting (6.17) into (6.21) and using (6.10), one finally finds that the first-order corrections to the eigenvalues are given by

$$h^{(1)}_{\sigma\mathbf{q}_n}=C_{\mathbf{q}_n}\left[\frac{1}{L_{\mathbf{q}_n}}+\frac{b^{(+)}_{\sigma\mathbf{q}_n}+b^{(-)}_{\sigma\mathbf{q}_n}}{\delta_{\mathbf{q}_n}}\right] \tag{6.22}$$

where

$$\delta_{\mathbf{q}_n}=\int_{-d/2}^{d/2}\frac{dz}{\hat{q}_{nz}(z)} \tag{6.23a}$$

$$\int_{-d/2}^{d/2}\frac{dz}{\hat{q}_{nz}(z)}\frac{1}{c(z)}=\frac{\delta_{\mathbf{q}_n}}{C_{\mathbf{q}_n}} \tag{6.23b}$$

$$\int_{-d/2}^{d/2}\frac{dz}{\hat{q}_{nz}(z)}\frac{1}{l_{\mathbf{q}_n}(z)}=\frac{\delta_{\mathbf{q}_n}}{L_{\mathbf{q}_n}} \tag{6.23c}$$

In (6.23), $\delta_{\mathbf{q}_n}$ may be interpreted as the length of the curved ray that is described by the local wavevector (notice that $\delta_{\mathbf{q}_n}=d/\hat{q}_{nz}$ in equilibrium). Moreover, $C_{\mathbf{q}_n}$ and $L_{\mathbf{q}_n}$ are averages of the speed of sound and the attenuation length, respectively, along the ray.

Equations (6.15) and (6.22) determine the eigenvalues $h_{\sigma\mathbf{q}_n}$ up to first order. To obtain the corresponding eigenfunctions, one has to compute the constants $A_{\sigma\mathbf{q}_n}$, $B_{\sigma\mathbf{q}_n}$,... to zeroth order from the $8\times 8$ system. Moreover, to zeroth order, the relations (3.9) and (3.10) for the adjoint eigenvectors apply also in the presence of a heat flux. This implies, in particular, that the normalization is still given by (3.11). In the next section I shall present the results.

## 7. RESULTS

In the previous sections I have discussed how the sound modes in the presence of a heat flux and sound-absorbing walls are evaluated. In this section I present the results. First, however, I recall the conditions under which they apply.

In the calculation I have assumed that the dimensionless parameters $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, and $\varepsilon_3'$ given by (5.4), (4.17), (5.5), and (6.6), respectively, are small. In view of (6.15), one may rephrase these conditions in terms of the local wavevectors as follows:

$$\varepsilon_1 = \frac{\Gamma(z)\, q_n(z)}{c(z)} \ll 1 \tag{7.1a}$$

$$\varepsilon_2 = \frac{1}{n\pi} \ll 1 \tag{7.1b}$$

$$\varepsilon_3 = \frac{1}{L_\nabla(z)\, q_n(z)} \ll 1, \qquad \varepsilon_3' = \frac{1}{L_\nabla(z)\, q_n(z)} \left(\frac{q_n(z)}{q_{nz}(z)}\right)^2 \ll 1 \tag{7.1c}$$

where, in the last member, I have used that $\Lambda = iq_{nz}$ to leading order. Notice that the smallness of $\varepsilon_3'$, which was used in order to apply the WKB method, implies the smallness of $\varepsilon_3$, and is thus a stronger condition. The first condition, (7.1a), yields an upper bound for the local wavevectors $\mathbf{q}_n(z)$, restricting them to the hydrodynamic regime, as mentioned before. The other two conditions, (7.1b) and (7.1c), basically imply lower bounds for $q_{nz}(z)$.

The eigenvalues have been calculated up to first order in the $\varepsilon$, while the eigenvectors have been evaluated to zeroth order. Yet, first-order corrections in the complex wavenumbers $Q_{z,\sigma\mathbf{q}_n}(z)$ [cf. (6.17)] have been kept in order to describe correctly the slow modulations in the amplitudes of the eigenfunctions.

The eigenvalues and eigenfunctions are most conveniently expressed in terms of the local, complex wavenumbers $Q_{z,\sigma\mathbf{q}_n}(z)$ introduced in (6.17). Inserting (6.22) into this expression, one finds

$$Q_{z,\sigma\mathbf{q}_n}(z)=q_{nz}(z)+i\sigma\left[\frac{1}{l_{\mathbf{q}_n}(z)}-\frac{C_{\mathbf{q}_n}}{c(z)}\frac{1}{L_{\mathbf{q}_n}}\right]\frac{1}{\hat{q}_{nz}(z)}$$
$$-i\sigma\frac{C_{\mathbf{q}_n}}{c(z)}\frac{b^{(+)}_{\sigma\mathbf{q}_n}+b^{(-)}_{\sigma\mathbf{q}_n}}{\delta_{\mathbf{q}_n}\hat{q}_{nz}(z)} \tag{7.2}$$

The quantities appearing in (7.2) have all been defined in Section 6. Consistent up to the order considered here, the eigenvalues can then be written in the form

$$h_{\sigma\mathbf{q}_n}=i\sigma c(z)\,Q_{\sigma\mathbf{q}_n}(z)+\tfrac{1}{2}\Gamma_s(z)\,Q^2_{\sigma\mathbf{q}_n}(z) \tag{7.3}$$

where

$$Q_{\sigma\mathbf{q}_n}(z)=(q^2_{\|}+Q^2_{z,\sigma\mathbf{q}_n})^{1/2} \tag{7.4}$$

Notice that $Q_{\sigma\mathbf{q}_n}(z)$ is defined in such a way that the right-hand side of (7.3) does not depend on $z$, as it should be.

The eigenvectors $\mathbf{a}_{\sigma\mathbf{q}_n}(\mathbf{r})$ and their adjoints $\bar{\mathbf{a}}_{\sigma\mathbf{q}_n}(\mathbf{r})$ can both be expressed in terms of the three eigenfunctions $\hat{p}_{\sigma\mathbf{q}_n}(z)$, $\hat{\phi}_{\sigma\mathbf{q}_n}(z)$, and $\hat{s}_{\sigma\mathbf{q}_n}(z)$ according to (3.2), (3.3), (3.6), (3.9), and (3.10). These will be given next.

Putting

$$N_{\mathbf{q}_n}(z)=\frac{1}{2\pi}\left[\frac{1}{\hat{q}_{nz}(z)\,\delta_{\mathbf{q}_n}}\frac{C_{\mathbf{q}_n}}{c(z)}\right]^{1/2} \tag{7.5}$$

one finds for the pressure eigenfunctions

$$\hat{p}_{\sigma\mathbf{q}_n}(z)=N_{\mathbf{q}_n}(z)\cos\left[\int_{-d/2}^{z}dz'\,Q_{z,\sigma\mathbf{q}_n}(z')+i\sigma b^{(-)}_{\sigma\mathbf{q}_n}\right] \tag{7.6}$$

Furthermore, the entropy eigenfunctions, which are localized in the boundary layers only, can be written as

$$\hat{s}_{\sigma\mathbf{q}_n}(z)=\hat{s}^{(+)}_{\sigma\mathbf{q}_n}(z)+\hat{s}^{(-)}_{\sigma\mathbf{q}_n}(z)$$

where

$$\hat{s}^{(+)}_{\sigma\mathbf{q}_n}(z)=-(-1)^n N^{(+)}_{\mathbf{q}_n}(\gamma^{(+)}-1)^{1/2}(\cosh b^{(+)}_{\sigma\mathbf{q}_n})$$
$$\times\exp\left[(1-i\sigma)\left(\frac{c^{(+)}q^{(+)}_n}{2D^{(+)}_T}\right)^{1/2}\left(z-\frac{d}{2}\right)\right] \tag{7.7a}$$

$$\hat{s}^{(-)}_{\sigma\mathbf{q}_n}(z)=-N^{(-)}_{\mathbf{q}_n}(\gamma^{(-)}-1)^{1/2}(\cosh b^{(-)}_{\sigma\mathbf{q}_n})$$
$$\times\exp\left[-(1-i\sigma)\left(\frac{c^{(-)}q^{(-)}_n}{2D^{(-)}_T}\right)^{1/2}\left(z+\frac{d}{2}\right)\right] \tag{7.7b}$$

Finally, the longitudinal velocity eigenfunctions are given by

$$\hat{\phi}_{\sigma \mathbf{q}_n}(z) = -\frac{i\sigma}{q_n(z)}\left[\hat{p}_{\sigma \mathbf{q}_n}(z) + \hat{\theta}_{\sigma \mathbf{q}_n}(z) + i\,\frac{(\gamma^{(+)}-1)^{1/2} D_T^{(+)} q_n^{(+)}}{c^{(+)}}\,\hat{s}_{\sigma \mathbf{q}_n}^{(+)}(z)\right.$$
$$\left. + i\,\frac{(\gamma^{(-)}-1)^{1/2} D_T^{(-)} q_n^{(-)}}{c^{(-)}}\,\hat{s}_{\sigma \mathbf{q}_n}^{(-)}(z)\right] \tag{7.8}$$

where

$$\theta_{\sigma \mathbf{q}_n}(z) = -N_{\mathbf{q}_n}(z)\left\{(-1)^n\left(\frac{q_{nz}(z)}{q_{nz}^{(+)}}\right)^{1/2}(\cosh b_{\sigma \mathbf{q}_n}^{(+)})\,\frac{\sinh q_{\|}(z+d/2)}{\sinh q_{\|} d}\right.$$
$$\left. - \left(\frac{q_{nz}(z)}{q_{nz}^{(-)}}\right)^{1/2}(\cosh b_{\sigma \mathbf{q}_n}^{(-)})\,\frac{\sinh q_{\|}(z-d/2)}{\sinh q_{\|} d}\right\} \tag{7.9}$$

Concerning Eq. (7.8), it should be noted that the entropy terms are small. However, I kept them since they become large when derivatives of $\hat{\phi}_{\sigma \mathbf{q}_n}$ with respect to $z$ are considered.

## 8. DISCUSSION

I conclude with a discussion of the results. The expressions for the sound modes that have been derived in this paper, and summarized in the last section, involve finite-size as well as nonequilibrium effects. Both effects will be discussed separately.

### 8.1. Finite-Size Effects

The finite-size effects are basically the same whether or not a heat flux is present. I shall therefore discuss these for the equilibrium case only. From (7.2)–(7.4) one then finds

$$Q_{z,\sigma \mathbf{q}_n} = q_{nz} - 2i\sigma\,\frac{b_{\sigma \mathbf{q}_n}}{d} \tag{8.1}$$

and

$$h_{\sigma \mathbf{q}_n} = i\sigma c q_n + \frac{1}{2}\Gamma_s q_n^2 + 2c\hat{q}_{nz}\,\frac{b_{\sigma \mathbf{q}_n}}{d} \tag{8.2}$$

where $q_{nz} = n\pi/d$. Comparing (8.2) with (2.17), it is observed that the eigenvalues in the finite system contain an extra term proportional to the wall-absorption coefficient $b_{\sigma \mathbf{q}_n}$. For finite values of $\beta$ (the acoustic admittance), this term can be shown to have a positive real part. Hence, it describes a sound damping that is due to the walls. Notice that the wall damping is of the same order as the damping in the fluid when $l_{\mathbf{q}_n}$ (the sound attenuation length) is of the same order as $d$.

To focus more on the role of $\beta$, let us choose $q_{nz} \gg q_{\|}$. In this case the expression (6.19) for the wall-absorption coefficient simplifies to

$$b_{\sigma \mathbf{q}_n} = \begin{cases} \dfrac{1}{2} \ln \dfrac{1+\beta}{1-\beta} & (\beta < 1) \\ \dfrac{1}{2} \ln \dfrac{\beta+1}{\beta-1} - i\sigma \dfrac{\pi}{2} & (\beta > 1) \end{cases} \tag{8.3}$$

In going from (6.19) to (8.3), I have replaced $\bar{\beta}_{\sigma \mathbf{q}_n}$ by $\beta$, thus dropping the small correction term appearing in (6.20). This is not allowed, however, when $\beta$ is very close to 1, since the expression (8.3) diverges logarithmically in that case. Actually, $b_{\sigma \mathbf{q}_n}$ goes through a resonance at $\beta = 1$. In the resonance one finds from (6.19) and (6.20) that

$$b_{\sigma \mathbf{q}_n} = \frac{1}{2} \ln \frac{2c}{\gamma \Gamma_l q_n} - i\sigma \frac{\pi}{4} \qquad (\beta = 1) \tag{8.4}$$

Note that a finite value for $b_{\sigma \mathbf{q}_n}$ at $\beta = 1$ has been obtained because I kept the full pressure tensor in the boundary condition (3.1c) instead of just the pressure, as is usually done.[6] Yet the expression (8.4) is probably only qualitatively correct, since I have still neglected terms of the same order in going from (4.14) to (6.19). Equation (8.4) indicates, however, that the wall-absorption coefficients are finite for all $\beta$, and it indicates the right order of magnitude. A complete discussion of the modes near the resonance requires further investigation. However, not too close to the resonance, the expression (8.3) is appropriate.

Using (8.3), I next discuss the properties of the equilibrium sound modes (with $q_{nz} \gg q_{\|}$) for various $\beta$. For completely rigid walls ($\beta = 0$) one finds $b_{\sigma \mathbf{q}_n} = 0$. Hence the eigenvalues (8.2) are the same as for an unbounded fluid, except that only discrete wavevectors are allowed. Notice from (7.7) and (7.9), however, that the entropy $\hat{s}_{\sigma \mathbf{q}_n}(z)$ in the boundary layers and the function $\hat{\theta}_{\sigma \mathbf{q}_n}(z)$ (which are proportional to $\cosh b_{\sigma \mathbf{q}_n}$) do not vanish for rigid walls. As $\beta$ is increased, $b_{\sigma \mathbf{q}_n}$ grows also, and the amplitudes of $\hat{s}_{\sigma \mathbf{q}_n}(z)$ and $\hat{\Theta}_{\sigma \mathbf{q}_n}(z)$ increase as well. At the resonance ($\beta = 1$), $b_{\sigma \mathbf{q}_n}(z)$ reaches a maximum and a phase shift occurs. If $\beta$ is increased even further, $\operatorname{Re} b_{\sigma \mathbf{q}_n}$ gets smaller, implying less wall absorption, and the amplitudes of $\hat{s}_{\sigma \mathbf{q}_n}(z)$ and $\hat{\theta}_{\sigma \mathbf{q}_n}(z)$ decrease. Finally, in the limit of completely elastic walls ($\beta = \infty$), one obtains $b_{\sigma \mathbf{q}_n} = -i\sigma\pi/2$, so that the wall damping in (8.2) vanishes again. Moreover, $\hat{s}_{\sigma\mathbf{q}_n}(z)$ and $\hat{\theta}_{\sigma\mathbf{q}_n}(z)$ completely disappear in this case.

I finally comment on the complex wavenumbers $Q_{z,\sigma\mathbf{q}_n}$, given in (8.1), which appear in the pressure eigenfunctions $\hat{p}_{\sigma\mathbf{q}_n}(z)$. Since the imaginary part of $Q_{z,\sigma\mathbf{q}_n}$ is small compared to $q_{nz}$, it follows that it gives rise to a large-scale modulation, whose effect is mostly pronounced near the resonance $\beta = 1$. It is illuminating to evaluate $\hat{p}_{\sigma\mathbf{q}_n}(z)$ for this case in the central region ($z \approx 0$) and close to the walls separately. Using (6.21) and (8.1), one obtains from (7.6) for values $z \approx 0$

$$\hat{p}_{\sigma\mathbf{q}_n}(z) = N_{\mathbf{q}_n} \cos\left(q_{nz}z + n\frac{\pi}{2}\right) \qquad (z \approx 0) \tag{8.5a}$$

and for $z \to \pm d/2$

$$\hat{p}_{\sigma\mathbf{q}_n}(z) = \frac{N_{\mathbf{q}_n}}{2}(\exp b_{\sigma\mathbf{q}_n}) \exp\left[\pm i\sigma\left(q_{nz}z + n\frac{\pi}{2}\right)\right] \qquad \left(z \to \pm\frac{d}{2}\right) \tag{8.5b}$$

Notice that these formulas apply on a scale that is intermediate between $q_{nz}^{-1}$ and $d$. It is observed from (8.5) and (8.2) that the corresponding pressure waves, given by

$$p_{\sigma\mathbf{q}_n}(\mathbf{r}, t) = \hat{p}_{\sigma\mathbf{q}_n}(z) \exp(-h_{\sigma\mathbf{q}_n}t)$$

(cf. Section 2) slowly change from standing waves at the center to propagating waves near the walls. The standing wave pattern near the center may be regarded as a superposition of two propagating waves that travel in opposite directions, both components having the same amplitude. As one approaches one of the walls, the amplitudes of the two propagating components become more and more unequal until, close to the wall, the component that propagates away from the wall is completely suppressed. In other words, for $\beta = 1$ there is no reflection of sound at the walls.

## 8.2. Nonequilibrium Effects

I now discuss how the sound modes are modified in the bulk fluid due to the presence of the heat flux. To this end, it is most appropriate to consider again a composite pressure wave, defined by

$$p_{\sigma\mathbf{q}_n}(\mathbf{r}, t) = p_{\sigma\mathbf{q}_n}(\mathbf{r}) \exp(-h_{\sigma\mathbf{q}_n}t) \tag{8.6}$$

where

$$p_{\sigma\mathbf{q}_n}(\mathbf{r}) = \hat{p}_{\sigma\mathbf{q}_n}(z) \exp(i\mathbf{q}_{||} \cdot \mathbf{r}_{||})$$

Moreover, to focus on the bulk effects, I restrict consideration in the following to completely rigid walls, so that the wall absorption coefficients vanish. Using (7.6), decompose (8.6) into two propagating parts,

$$p_{\sigma \mathbf{q}_n}(\mathbf{r}, t) = p^{+}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) + p^{-}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) \tag{8.7}$$

where

$$p^{\pm}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) = N_{\mathbf{q}_n}(z) \exp i\phi^{\pm}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) \tag{8.8}$$

and $\phi^{\pm}_{\sigma \mathbf{q}_n}(\mathbf{r}, t)$ are complex phase functions, given by

$$\phi^{\pm}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) = \pm \int_{-d/2}^{z} dz'\, Q_{z,\sigma \mathbf{q}_n}(z') + \mathbf{q}_{||} \cdot \mathbf{r}_{||} + ih_{\sigma \mathbf{q}_n} t \tag{8.9}$$

The relations $\operatorname{Re} \phi^{\pm}_{\sigma \mathbf{q}_n}(\mathbf{r}, t) = 0$ determine the locations of the wavefronts, denoted by $\mathbf{r}^{\pm}_{\sigma \mathbf{q}_n}(t)$. Using (7.2) and (7.3), one finds from (8.9) that

$$\dot{\mathbf{r}}^{\pm}_{\sigma \mathbf{q}_n} = \sigma c(z)\, \hat{\mathbf{q}}^{\pm}_n(z) \tag{8.10}$$

where $\hat{\mathbf{q}}^{\pm}_n(z) = [\mathbf{q}_{||} \pm q_{nz}(z)\, \mathbf{e}_z]/q_n(z)$. Equation (8.10) implies that the phase velocities are not constant (in magnitude or in direction). In fact, (8.10) describes the bending of sound due to the spatially inhomogeneous speed of sound.

Denoting by $s$ the distance along the curved sound rays that run perpendicular to the wavefronts, one finds from (8.10) and (6.12), (6.15) that

$$\begin{aligned} \frac{d}{ds} \mathbf{r}^{\pm}_{\sigma \mathbf{q}_n} &= \sigma \hat{\mathbf{q}}^{\pm}_n \\ \frac{d}{ds} \hat{\mathbf{q}}^{\pm}_n &= -(\mathbf{1} - \hat{\mathbf{q}}^{\pm}_n \hat{\mathbf{q}}^{\pm}_n) \cdot \mathbf{e}_z \frac{d}{dz} \ln c \end{aligned} \tag{8.11}$$

These equations determine the rays uniquely, given some "initial" conditions at $s = 0$. Equations (8.11) may also be derived from a Hamiltonian.[10]

Finally, I comment on the anomalous sound damping. To focus on the simplest case, I assume that $c$ is independent of temperature, so that there is no sound bending. Moreover, I put $\mathbf{q}_{||} = 0$ and consider only the wave $p^{+}_{+\mathbf{q}_n}(\mathbf{r}, t)$ which propagates in the positive $z$ direction, the wavefront being located at $\mathbf{r}(t) = (ct - d/2)\, \mathbf{e}_z$. Inserting (7.2)–(7.5) into (8.8), (8.9), one obtains

$$\begin{aligned} p^{+}_{+\mathbf{q}_n}(t) \equiv p^{+}_{+\mathbf{q}_n}(\mathbf{r}(t), t) &= \frac{1}{2\pi \sqrt{d}} \exp\left[ -\int_{-d/2}^{z(t)} \frac{dz'}{l_{\mathbf{q}_n}(z')} \right] \\ &= \frac{1}{2\pi \sqrt{d}} \exp\left[ -\frac{q_{nz}^2}{2c} \int_{-d/2}^{z(t)} dz'\, \Gamma_s(z') \right] \end{aligned} \tag{8.12}$$

where also (6.18) has been used. The expression (8.12) describes the amplitude at the wavefront of a pressure wave that starts at time $t=0$ at the wall $z=-d/2$. It is observed from (8.12) that the wave is not exponentially damped in general. Rather, the damping depends in a functional way on the spatial profile of the sound damping coefficient.

To illustrate the implications of (8.12), let us assume, as an example, that the equation of state for $\Gamma_s$ is of the form $\Gamma_s(T)=\Gamma_{s0}\exp(-mT)$, where $m>0$.[11] Assuming, furthermore, that the thermal conductivity is constant, one obtains a linear temperature profile, so that

$$\Gamma_s(z)=\Gamma_s^{(-)}\exp\left[-m\frac{dT}{dz}\left(z+\frac{d}{2}\right)\right] \tag{8.13}$$

In this case it follows from (8.12) that the amplitude of the wave that arrives after the time $t=d/c$ at the wall $z=d/2$ is given by

$$p_{+\mathbf{q}_n}^{+}\left(t=\frac{d}{c}\right)\propto\exp\left[\frac{\exp[-m(dT/dz)d]-1}{l_{\mathbf{q}_n}^{(-)}m\,dT/dz}\right] \tag{8.14}$$

For small temperature gradients, $|dT/dz|\ll(md)^{-1}$, the right-hand side of (8.14) reduces to $\exp[-d/l_{\mathbf{q}_n}^{(-)}]$. This means that a wave starting at $z=-d/2$ with an attenuation length $l_{\mathbf{q}_n}^{(-)}<d$ is essentially damped out before it reaches the other wall. On the other hand, if $dT/dz\gg(md)^{-1}$, one finds from (8.14) that

$$p_{+\mathbf{q}_n}^{+}\left(t=\frac{d}{2}\right)\propto\exp\left[-\frac{1}{l_{\mathbf{q}_n}^{(-)}m\,dT/dz}\right] \tag{8.15}$$

independent of $d$! Notice that (8.15) is of the order 1 when

$$l_{\mathbf{q}_n}^{(-)}>\left(m\frac{dT}{dz}\right)^{-1}=\left|\frac{d\ln\Gamma_s}{dz}\right|^{-1}$$

This implies that a wave starting at $z=-d/2$ with a local attenuation length that is larger than the scale on which $\Gamma_s(z)$ changes effectively experiences so little damping that a substantial fraction of that wave arrives at the other wall, regardless of how large $d$ is. This extreme reduction of the damping occurs because the wave is propagating in the direction of decreasing $\Gamma_s(z)$. In the opposite case, i.e., when the wave propagates in the direction of increasing $\Gamma_s$, the effective damping is faster than exponential.

[11] This is actually a good fit for water.[(11)]

## ACKNOWLEDGMENT

Support by the Division of Sponsored Research at the University of Florida is gratefully acknowledged.

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